A NUMERICAL SOLUTION FOR THE STRESS DISTRIBUTION IN A ROTATING DISK

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SUMMARY

An approximate numerical solution procedure for calculating the plane stress distribution in a rotating disk is presented. The procedure involves numerical techniques applied to the relationships arising from the theory of elasticity, and is, in general, intended for solving two dimensional stress distributions in rotating disks.

An important feature of the method developed is the use of polar coordinates in conuunction with finite difference equations. Polar grid networks lend themselves especially well to rotating disk problems in that the grid may be selected such that it fits concentric circular boundaries exactly, thus avoiding approximations to certain boundary-adjacent points.

All necessary relationships for solving two-dimensional disk problems are presented. For solving the biharmonic equation, the "composite solution" is employed, and its general properties discussed. As an example of the applicability of the developed procedure, it was applied to the classical one-dimensional problem of the plane rotating disk with a centrally located circular hole. It was noted for a symmetrical stress distribution, the number of terms needed in the composite solution may be reduced considerably.

Relaxation techniques in conjunction with finite differences were programmed for the Burroughs 220 Algebraic Compiler, and high-speed digital computation was thereby performed to solve the biharmonic equation for the required stress functions of the "composite solution." A final computer

program was employed to combine the "composite" stress functions and to compute the stresses. The exact values of the stresses, obtained from the known solution, were also computed in the final program to furnish a means of comparison.

At the inner boundary of the disk, the tangential stress computed numerically is 11.32 per cent lower than the exact value for this same point. At other points the error is less, with the numerical and exact radial stresses agreeing almost exactly at the inner and outer boundaries. It is believed that the high rate of change of the tangential stress with respect to the radius in the vicinity of the inner boundary is the underlying reason for the magnitude of the error observed there. Errors in the final results are discussed with respect to rotational speed, stress gradients, and grid spacings.

CHAPTER I

INTRODUCTION

Due to an advancing state of the art, rotational speeds of certain equipment items, such as turbine wheels and the impellers of centrifugal compressors, have increased to a point where the induced stresses are highly critical. In applied design, the stress distribution in the body is usually computed by treating the problem as one of a rotating disk having a configuration which approximates that of the actual part. From this standpoint, a solution to the problem of the rotating disk in general would be highly desirable.

Insofar as stress distributions are concerned, there presently exists an exact solution for the one-dimensional rotating disk,[1] and several approximate methods for the two-dimensional case.[2,3,4] One of the two-dimensional methods has been extended to account for variable disk thickness.[5] A review of the two-dimensional solutions reveals that they are of two basic types: either closed form approximations to a specific problem, or applications of the small hole - infinite plate theory.

It was felt that a more general approach to the two-dimensional problem via the known theory of elasticity would fill an existing gap in the methods of solution now available. For this reason, the general relationships for a two-dimensional problem in polar coordinates were used to formulate a numerical, or finite difference, solution procedure. In that it was desired to check the validity and accuracy of the procedure developed, the procedure was applied directly to the one-dimensional rotating disk with a central hole, for which an exact solution is available.

The solution procedure presented is intended to be a general outline for numerical solutions of two-dimensional rotating disk problems.

The notation employed is the same as that in Reference 1.

CHAPTER II

ELASTIC THEORY IN POLAR COORDINATES FOR TWO-DIMENSIONAL PROBLEMS

Basic Relations

In considering a rotating disk of constant thickness, in which axial loads are absent, and the thickness relatively small compared to the diameter, it is reasonable to assume that axial stresses are negligible, thus, plane stress theory will be used. In the literature, this two-dimensional state of stress is referred to as "plane stress." Application of the conditions of plane stress to the general three-dimensional elasticity relations yields a particular set of equations which govern the state of plane stress.[1]

Equilibrium Conditions

For the condition of zero angular acceleration of the body under consideration, tangential body forces are zero, and dynamic equilibrium of the stresses is stated mathematically by

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0 \qquad (1)$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0 \qquad (2)$$

where R is a body force term which may be expressed in terms of a potential as:

$$R = -\frac{\partial V}{\partial r}$$

Strain Deformation Relations

The strains are defined in terms of the displacements as follows:

$$\epsilon_r = \frac{\partial u}{\partial r} \tag{3}$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{4}$$

$$\gamma_{r_\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{5}$$

Stress-Strain Relations

From Hooke's Law, the elastic stress-strain relations are given by:

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) \tag{6}$$

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \tag{7}$$

$$\gamma_{r_\theta} = \frac{2(1+\nu)}{E}\tau_{r_\theta} \tag{8}$$

Stress Function

In conjunction with the previous conditions, a stress function, ϕ, is proposed such that the stresses may be expressed as functions of ϕ and/or its derivatives plus a possible body force term. It is further required that the resulting expressions for the stresses satisfy the equilibrium equations (1) and (2). The stresses in terms of the stress function are:

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + V \tag{9}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} + V \tag{10}$$

$$\tau_{r_\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} \tag{11}$$

where V is assumed to be a function of radius only. That the above expressions for the stresses are valid may be verified by substituting them into the equilibrium equations.

Compatibility Equation

Examination of the strain-deformation equations (3), (4), and (5) reveals that the three strains are all expressed in terms of no more than the two displacements u and v. If the strains are to vary in a continuous manner over the body, they are not independent of one another, and are, in fact, related in some specific manner. Mathematically, this relationship between the strains is expressed as the "compatibility equation," which is obtained from equations (3), (4), and (5) (see Appendix A). The compatibility equation is

$$\left(\frac{\partial^2}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta} \right)(\gamma_{r_\theta}) + \left(\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\right)(\epsilon_r) - \left(r\frac{\partial^2}{\partial r^2} + 2\frac{\partial}{\partial r}\right)(\epsilon_\theta) = 0 \tag{12}$$

Rigorous mathematical investigations have revealed that the compatibility equation is a necessary condition for single-valued displacements. Furthermore, in a simply connected region, or in a simply connected subregion of a multiply connected region, it is a sufficient condition.

Boundary Conditions

Physically, a statement of the boundary conditions is simply that normal and shear stresses are zero on the boundaries of the body. An equivalent mathematical expression is obtained by considering an element of volume, of unit depth, bounded by Δx, Δy, and Δs, where Δs denotes a differential element of boundary. Such an element of volume is shown in Figure 1. For the sake of clarity, and relative ease of derivation, the boundary conditions will be derived in rectangular coordinates, and later transformed into polar coordinates.

Noting that normal and shear stresses are zero on face Δs, we have, from static equilibrium,

$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

For the x-summation, we have

$$-\sigma_x \Delta y + \tau_{xy} \Delta x + X \left(\frac{1}{2} \Delta x \Delta y\right) = 0$$

or,

$$-\sigma_x \left(\frac{\Delta y}{\Delta s} \right) + \tau_{xy} \left(\frac{\Delta x}{\Delta s} \right) + X \left(\frac{1}{2} \frac{\Delta x}{\Delta s} \right) \Delta y = 0$$

Taking the limit as Δx, Δy, and Δs all approach zero, the body force term approaches zero, and we have

$$-\sigma_x \frac{dy}{ds} + \sigma_{xy} \frac{dx}{ds} = 0 \qquad (13)$$

From the y-summation, we have

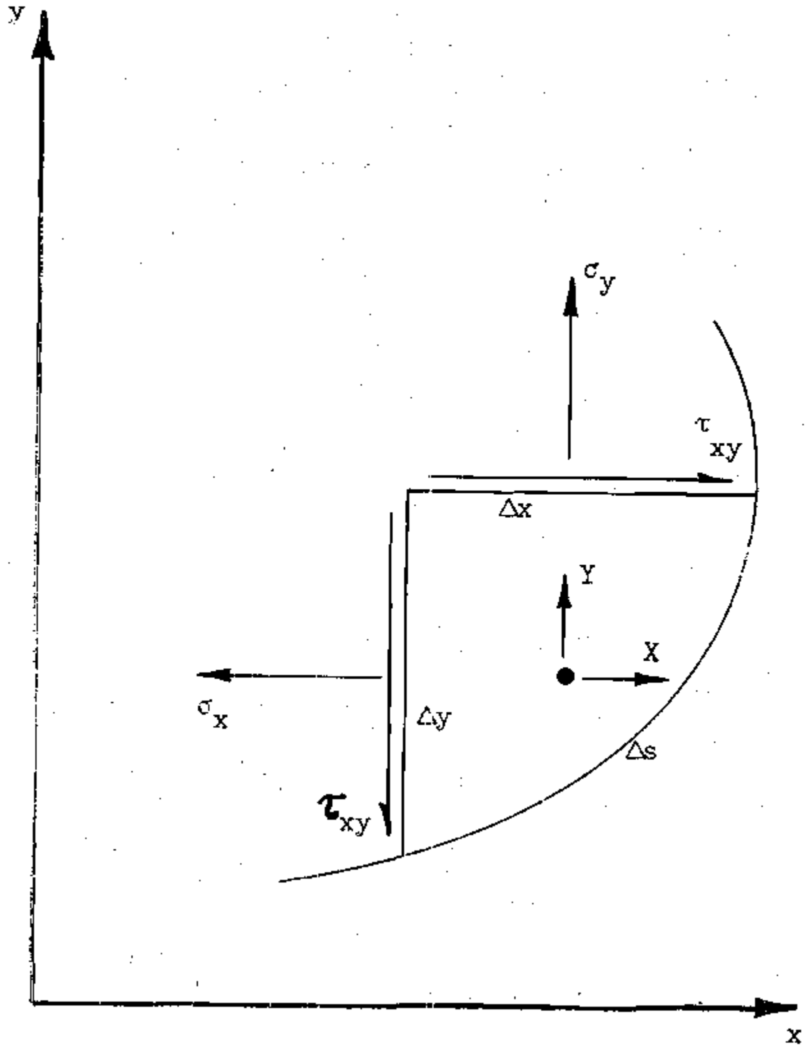


Figure 1. Element on Boundary.

$$\sigma_y \Delta x - \tau_{xy} \Delta y + Y \left(\frac{1}{2} \Delta x \Delta y \right) = 0$$

Taking the limit as above, we have

$$\sigma_y \frac{dx}{ds} - \tau_{xy} \frac{dy}{ds} = 0 \tag{14}$$

These two expressions obtained from the x and y-summations of the forces constitute the boundary conditions that must be satisfied by the stresses in order that the condition of zero boundary loading be valid.

<u>Michell's Displacement Conditions</u>

Any function $\phi(r,\theta)$ will yield a set of stresses which will satisfy the equilibrium equations provided these stresses are calculated using equations (9) to (11). By this technique, more than one set of stresses, corresponding to different functions ϕ, may be obtained which will satisfy the equations of equilibrium and a given set of boundary conditions, because the three stresses σ_r, σ_θ, $\tau_{r\theta}$ need satisfy only two equations of equilibrium. At this point, the problem is "statically indeterminate" in that an insufficient number of equations are available to define a unique state of stress. Additional conditions on ϕ are obtained by studying the components of displacement. The correct stress function is the one which results in displacements that are single-valued and continuous. As stated above, the compatibility equation is a necessary condition, and therefore, the correct stress function results in the compatibility equation being satisfied. However, in the case of a multiply connected region, the compatibility equation is not sufficient, and other conditions are required. These extra conditions are known as Michell's conditions and are given as:

$$\int_{C_i} \frac{\partial(\nabla^2\phi)}{\partial n}\, ds = -\,(1-\nu)\int_{C_i} \frac{\partial V}{\partial n}\, ds, \ (i = 1,2,3,\ldots,n) \tag{15}$$

$$\int_{C_i}\left[y\,\frac{\partial(\nabla^2\phi)}{\partial n} - x\,\frac{\partial(\nabla^2\phi)}{\partial s}\right] ds = -\,(1-\nu)\int_{C_i}\left[y\,\frac{\partial V}{\partial n} - x\,\frac{\partial V}{\partial s}\right] ds \tag{16}$$

$$+\,(1+\nu)\int_{C_i} (Vm)\, ds, \ (i = 1,2,3,\ldots,n)$$

$$\int_{C_i} y\,\frac{\partial(\nabla^2\phi)}{\partial s} + x\,\frac{\partial(\nabla^2\phi)}{\partial n}\ \ ds = -(1-\nu)\int_{C_i}\left[y\,\frac{\partial(\nabla^2\phi)}{\partial s} + \right. \tag{17}$$

$$\left.\frac{\partial(\nabla^2\phi)}{\partial n}\right] ds + (1+\nu)\int_{C_i} (V\ell)\, ds, \ (i = 1,2,3,\ldots,n)$$

for a multiply connected region with n internal boundaries. The line integrals indicated are taken around any closed contour, C_i, within the body which encloses an internal boundary. There exists a distinct set of three equations for each internal boundary of the body. It should be noted that the specific form of Michell's conditions stated above was arrived at by assuming that boundary loadings of an external nature and thermal stresses were both zero.

Equations (1) through (17) are now sufficient to define completely a state of plane stress consistent with the assumptions set forth in the development. Any solution ϕ that satisfies the above conditions is then a unique solution to the particular plane stress problem in which the body has no translational or rotational dislocations.

CHAPTER III

APPLICATION OF ELASTIC THEORY TO ROTATING DISK

Statement of Problem

It is now desired to apply the theory developed in the preceding chapter to the problem of a circular disk rotating about its geometrical center in plane motion.

Boundary Conditions

The boundaries of the disk are to be free of externally applied loads, and shall experience no manner of constraint. All loadings of the disk material shall be strictly due to the centrifugal body forces arising from the rotation of the disk. Thermal stresses are thereby neglected.

Description of the Disk

Since no specific engineering problem gave rise to this investigation, the choice of dimensions and specific geometry was deemed arbitrary, subject to practical considerations. At the outset, it was decided that the investigation would be restricted to disks of constant thickness, due to the large number of extra terms which must be included to account for variations in thickness. In that it was desired to test and evaluate the solution procedure developed, it was necessary to select a disk configuration for which an exact solution exists. To this end, the disk selected was one of eight inches outside diameter possessing a centrally located circular hole of two inches diameter. The disk material was considered to be steel in the computations, and the rotational speed fixed at 2000 rpm.

Potential Function

From equations (1), (2), (9), (10), and (11), we have the pair of conditions

$$R = -\frac{\partial V}{\partial r} = \rho\omega^2 r$$

$$\frac{1}{r}\frac{\partial V}{\partial \theta} = 0$$

Thus,

$$\frac{dV}{dr} = -\rho\omega^2 r$$

$$V = \frac{\rho\omega^2 r^2}{2}$$

with the arbitrary constant being taken as zero. With the above results, the stress equations may be rewritten in terms of this potential function as

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} - \frac{\rho\omega^2 r^2}{2} \tag{18}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} - \frac{\rho\omega^2 r^2}{2} \tag{19}$$

$$\tau_{r_\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} \tag{20}$$

Biharmonic Equation

In light of the properties of the compatibility equation (12), and the fact that the stress function expressions for the stresses satisfy the equilibrium equations (1) and (2), it would be desirable to combine the two results. The resulting equation would then incorporate the properties of single-valuedness and equilibrium into one mathematical expression.

Initially, the stress-strain relations (6) to (8) are substituted into the compatibility equation (12) in order to obtain an equation in terms of the stresses. The resulting expression, after substitution and rearrangement appears as

$$2(1+\nu)\left(\frac{\partial^2}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial}{\partial \theta}\right)(\tau_{r_\theta}) + (1+2\nu) \frac{\partial \sigma_r}{\partial r} - \frac{1\partial^2 \sigma_r}{r \partial \theta^2} + \nu r \frac{\partial^2 \sigma_r}{\partial r^2}$$

$$- (2 + \nu) \frac{\partial \sigma_\theta}{\partial r} + \frac{\nu}{r} \frac{\partial^2 \sigma_\theta}{\partial\ ^2} - r \frac{\partial^2 \sigma_\theta}{\partial r^2} = 0$$

The stresses are now rewritten in terms of the stress function using equations (18) through (20), and after rearranging, the equation appears in the form

$$\frac{\partial^4 \phi}{\partial r^4} + \frac{2}{r} \frac{\partial^3 \phi}{\partial r^3} - \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^3} \frac{\partial \phi}{\partial r} - \frac{2}{r^3} \frac{\partial^3 \phi}{\partial r \partial \theta^2} \tag{21}$$

$$+ \frac{2}{r^2} \frac{\partial^4 \phi}{\partial r^2 \partial \theta^2} + \frac{4}{r^4} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^4} \frac{\partial^4 \phi}{\partial \theta^4} = 2(1-\nu)\ \rho\omega^2$$

which is known as the non-homogensous biharmonic equation. The motivation for this terminology becomes evident when it is realized that equation (21) may be factored into a form

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 \phi = 2(1-\nu)\ \rho\omega^2$$

which is simply one Laplacian operator acting upon another, or using the notation ∇^2 for the Laplacian operator, we have the following notation for the nonhomogeneous biharmonic

$$\nabla^4\phi = \nabla^2 \cdot \nabla^2\phi = 2(1 - \nu)\ \rho\omega^2$$

If the right hand side of (21) is zero, then the partial differential equation is homogeneous, and is merely the Laplacian of a harmonic function taken twice, or biharmonic.

Due to the features included in its derivation, the biharmonic equation, in conjunction with a given set of boundary conditions, will yield a stress function that will satisfy equilibrium, and will ensure single-valued displacements in a simply connected region, or a simply connected sub region of a multiply connected region. To ensure that the displacements are single valued over a multiply connected region, such as the disk under consideration, Michell's displacement conditions must be satisfied as well as the biharmonic equation.[6]

The solution of this particular problem will then be a specific stress function ϕ that will satisfy equations (15), (16), (17), and (21). From this stress function, the stresses σ_r, σ_θ, τ_{r_θ} are easily obtained by applying equations (18) through (20).

Boundary Conditions

The biharmonic equation (21) has the form

$$\frac{\partial^4\phi}{\partial x^4} + 2\,\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = -\ (1-\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

in rectangular coordinates, where

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} + V$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} + V$$

$$\tau_{xy} = \frac{\partial^2\phi}{\partial x\partial y}$$

Application of boundary conditions is made by substitution of the above expressions for the stresses into equations (13) and (14). From equation (13), we have, on the boundaries, i,

$$-\left(\frac{\partial^2\phi}{\partial y^2} + V\right)\frac{dy}{ds} - \left(\frac{\partial^2\phi}{\partial x\partial y}\right)\frac{dx}{ds} = 0$$

or,

$$\frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right) = -V\frac{dy}{ds}$$

This leads to

$$\left.\frac{\partial\phi}{\partial y}\right)_i = -\int_o^s Vdy + b_i$$

By substitution into (14), we obtain

$$\left(\frac{\partial^2\phi}{\partial x^2}\right)\frac{dx}{ds} + \left(\frac{\partial^2\phi}{\partial x\partial y}\right)\frac{dy}{ds} + V\frac{dx}{ds} = 0$$

or

$$\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right) = -V\frac{dx}{ds}$$

which leads to

$$\left.\frac{\partial\phi}{\partial x}\right)_i = - \int_o^s Vdx + a_i$$

For the nonhomogeneous biharmonic equation,

$$V = - \frac{\rho\omega^2}{2} (x^2 + y^2)$$

and we have from the above integrals

$$\left.\frac{\partial\phi}{\partial x}\right)_i = \frac{\rho\omega^2}{2} \int_o^s (x^2 + y^2)\, dx + a_i$$

$$\frac{\partial\phi}{\partial y}_i = \frac{\rho\omega^2}{2} \int_o^s (x^2 + y^2)\, dy + b_i$$

On the boundaries 1 and 2, the inner and outer boundaries, respectively we have

$$x^2 + y^2 = R_i^{\,2}$$

By substitution, we have

$$\left.\frac{\partial\phi}{\partial x}\right)_i = \frac{\rho\omega^2}{2} \int_o^s R_i^{\,2} dx + a_i$$

$$\left.\frac{\partial\phi}{\partial y}\right)_i = \frac{\rho\omega^2}{2} \int_o^s R_i^{\,2} dy + b_i$$

or,

$$\left.\frac{\partial\phi}{\partial x}\right)_i = \frac{\rho\omega^2}{2} R_i^{\,2} x + a_i$$

$$\frac{\partial\phi}{\partial y}_i = \frac{\rho\omega^2}{2} R_i^{\,2}\, y + b_i$$

Furthermore,

$$\phi_i = \int_o^s \left[\left(\frac{\partial\phi}{\partial x}\right)_i dx + \left(\frac{\partial\phi}{\partial y}\right)_i dy \right] + C_i$$

which leads directly to

$$\phi_i = \frac{\rho\omega^2}{2} R_i^{\,2} \left(\frac{x^2}{2} + \frac{y^2}{2}\right) + a_i x + b_i y + C_i$$

Thus, on these two concentric circular boundaries, ϕ has the form

$$\phi_i = \frac{\rho\omega^2}{2} R_i^{\,4} + A_i x + b_i y + C_i \tag{22}$$

Since the differential equation and the boundary conditions are both linear, we may separate the solution into two parts: a complementary part, and a particular part.

For the related homogeneous equation, simply set V equal to zero, and we have

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2}$$

$$\zeta_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

and

$$\nabla^4\phi = 0$$

When the above expressions for the stresses are used in conjunction with equations (13) and (14), the stress function and its derivatives have, on the boundaries, the form

$$\left.\frac{\partial\phi}{\partial x}\right)_i = a_i$$

$$\left.\frac{\partial\phi}{\partial y}\right)_i = b_i$$

$$\phi_i = a_i x + b_i y + C_i$$

It is now clear that the solution to the nonhomogeneous biharmonic equation with boundary conditions (22) may be written as the sum of a complementary (homogeneous) solution, and a particular solution. The complementary solution satisfies the homogeneous biharmonic and the boundary conditions:

$$\phi_i = a_i x + b_i y + C_i$$

and the particular solution satisfies the nonhomogeneous biharmonic and the boundary conditions:

$$\phi_i = \frac{\rho\omega^2}{4} R_i^4$$

Consistent with the preceding remarks, let the complete solution of the nonhomogeneous biharmonic equation be

$$\phi = \phi^* + \phi_d$$

where ϕ^* is the complementary solution, and ϕ_d is the particular solution. From this, it follows that on a boundary i for this problem,

$$\phi_i^* = a_i x + b_i y + c_i \tag{23}$$

$$\phi_{d_i} = \frac{\rho\omega^2}{4} R_i^4 \tag{24}$$

The constants a_i, b_i, c_i will, in general, have different values for each boundary i, but may be chosen as arbitrarily zero on one of these boundaries. From previous results applied to the ith boundary, we have

$$\left.\frac{\partial\phi^*}{\partial x}\right)_i = a_i \tag{25}$$

$$\left.\frac{\partial\phi^*}{\partial y}\right)_i = b_i \tag{26}$$

$$\left.\frac{\partial\phi}{\partial x} d\right)_i = \frac{\rho\omega^2}{2} R_i^2 x \tag{27}$$

$$\left.\frac{\partial\phi}{\partial y} d\right)_i = \frac{\rho\omega^2}{2} R_i^2 y \tag{28}$$

Composite Solution

Consider a body defined by $n + 1$ boundaries, and having a_i, b_i, and c_i arbitrarily zero on not more than one of those boundaries. A composite solution is proposed that will satisfy the boundary values (23) through (28). This composite solution consists of a particular solution plus a summation of complementary solutions.

From relations (23), (25), and (26), it may be seen that the complementary solution must satisfy three conditions on each boundary with the exception of one which is arbitrary, or a total of 3n conditions. The composite complementary solution is composed of 3n solutions, each of which will satisfy one of the 3n boundary conditions while not conflicting with the others.[7] This composite complementary solution will then have the form

$$\phi^* = \sum_{i=1}^{n} (a_i\phi_{a_i} + b_i\phi_{b_i} + c_i\phi_{c_i})$$

Furthermore, on any boundary, i,

$$\phi_{a_i}\Big)_i = x \qquad \frac{\partial\phi_{a_i}}{\partial x}\Big)_i = 1 \qquad \frac{\partial\phi_{a_i}}{\partial y} = 0$$

$$\phi_{b_i}\Big)_i = y \qquad \frac{\partial\phi_{b_i}}{\partial x}\Big) = 0 \qquad \frac{\partial\phi_{b_i}}{\partial y} = 1$$

$$\phi_{c_i}\Big)_i = 1 \qquad \frac{\partial\phi_{c_i}}{\partial x}\Big) = 0 \qquad \frac{\partial\phi_{c_i}}{\partial y} = 0$$

and on any other boundary, q, where $q \neq i$, we have the following properties:

$$\phi_{a_i}\Big)_q = \frac{\partial\phi_{a_i}}{\partial x}\Big)_q = \frac{\partial\phi_{a_i}}{\partial y}\Big)_q = 0$$

$$\phi_{b_i}\Big)_q = \frac{\partial\phi_{b_i}}{\partial x}\Big)_q = \frac{\partial\phi_{b_i}}{\partial y}\Big)_q = 0$$

$$\phi_{c_i}\Big)_q = \frac{\partial\phi_{c_i}}{\partial x}\Big)_q = \frac{\partial\phi_{c_i}}{\partial y}\Big)_q = 0$$

Since

$$\left.\frac{\partial\phi^*}{\partial x}\right)_i = \sum_{i=1}^{n} a_i\left(\frac{\partial\phi_{a_i}}{\partial x}\right)_i + b_i\left(\frac{\partial\phi_{b_i}}{\partial x}\right)_i + c_i\left(\frac{\partial\phi_{c_i}}{\partial x}\right)_i$$

$$\left.\frac{\partial\phi^*}{\partial y}\right)_i = \sum_{i=1}^{n} a_i\left(\frac{\partial\phi_{a_i}}{\partial y}\right)_i + b_i\left(\frac{\partial\phi_{b_i}}{\partial y}\right)_i + c_i\left(\frac{\partial\phi_{c_i}}{\partial y}\right)_i$$

it is easily seen that conditions (30) and (31) are equivalent to conditions (23), (25), and (26). Thus, the composite complementary solution will satisfy the necessary boundary conditions, and hence, is valid. In order to obtain the complete composite solution, it is merely required that the particular solution, ϕ_d, be added to the complementary solution, ϕ^*, or

$$\phi = \sum_{i=1}^{n} (a_i\phi_{a_i} + b_i\phi_{b_i} + c_i\phi_{c_i}) + \phi_d \tag{32}$$

Since the disk under consideration is defined by two boundaries, it follows that

$$n + 1 = 2$$

or,

$$n = 1$$

The composite solution for this particular disk then assumes the form

$$\phi = \sum_{i=1}^{1} (a_i\phi_{a_i} + b_i\phi_{b_i} + c_i\phi_{c_i}) + \phi_d$$

or, in expanded form

$$\phi = a_1\phi_{a_1} + b_1\phi_{b_1} + c_1\phi_{c_1} + \phi_d \tag{33}$$

Let the boundaries 1 and 2 be the boundaries of the central hole and the outer edge of the disk, respectively. On boundary 2, ϕ_{a_1}, ϕ_{b_1}, ϕ_{c_1}, and their derivatives will all be equal to zero.

<u>Michell's Conditions for Composite Solution</u>

Assuming that some method exists for obtaining the values of the various stress functions in the composite solution, there still remains the problem of evaluating the multiplicity of associated constants. The remaining set of conditions which may be employed to this end are Michell's displacement conditions.

In order to make the presentation less cumbersome, the following operator notation will be employed:

$$I_1\{\ \}_i = \int_{c_i} \frac{\partial}{\partial n}\, ds$$

$$I_2\{\ \}_i = \int_{c_i} \left[y\,\frac{\partial}{\partial n} - \frac{\partial}{\partial s}\right] ds$$

$$I_3\{\ \}_i = \int_{c_i} \left[y\,\frac{\partial}{\partial s} + x\,\frac{\partial}{\partial n}\right] ds$$

$$I_4\{\ \}_i = \int_{c_i} ds$$

Since the differentiation operator is linear,

$$\nabla^2\phi = a_1\nabla^2\phi_{a_1} + b_1\nabla^2\phi_{b_1} + c_1\nabla^2\phi_{c_1} + \nabla^2\phi_d$$

and equations (15) through (17) assume the form

$$a_1 I_1\left\{\nabla^2\phi_{a_1}\right\}_1 + b_1 I_1\left\{\nabla^2\phi_{b_1}\right\}_1 + c_1 I_1\left\{\nabla^2\phi_{c_1}\right\}_1 = \tag{34}$$

$$- I_1\left\{\nabla^2\phi_d\right\}_1 - (1 - \nu)\, I_1\left\{V\right\}_1$$

$$a_1 I_2\left\{\nabla^2\phi_{a_1}\right\}_1 + b_1 I_2\left\{\nabla^2\phi_{b_1}\right\}_1 + c_1 I_2\left\{\nabla^2\phi_{c_1}\right\}_1 = \tag{35}$$

$$- I_2\left\{\nabla^2\phi_d\right\}_1 - (1 - \nu) I_2\left\{V\right\}_1 + (1 + \nu) I_4\left\{Vm\right\}_1$$

$$a_1 I_3\left\{\nabla^2\phi_{a_1}\right\}_1 + b_1 I_3\left\{\nabla^2\phi_{b_1}\right\}_1 + c_1 I_3\left\{\nabla^2\phi_{c_1}\right\}_1 = \tag{36}$$

$$- I_3\left\{\nabla^2\phi_d\right\}_1 - (1 - \nu) I_3\left\{V\right\}_1 + (1 + \nu) I_4\left\{V\ell\right\}_1$$

Assuming that the composite stress functions may be found, this problem has been reduced to one of three equations and three unknowns, thus the solution may be found.

When the constants have been evaluated, the stress function ϕ may be evaluated in turn. Substituting this stress function into equation (9) through (11) will yield the stress distribution throughout the disk, and the solution may be considered as complete.

CHAPTER IV

SPECIAL FEATURES OF COMPOSITE SOLUTION

Properties of Solutions

If the body under consideration is symmetric with no external loads, and symmetrically distributed internal loads, it will possess a stress distribution. For simplicity, consider the case where the symmetry is such that the stress is the same in each quadrant.

Stating that the stress distribution is the same in each quadrant implies that the stresses must be even functions with respect to the variables x and y.

Noting that the x and y stresses are expressed in cartesian coordinates as the second partial derivatives with respect to y and x respectively, it is possible to reach certain conclusions about the properties of the stress function. Recalling that if one has a function of two variables, say x and y, the odd and even properties of that function with respect to either variable will change if a partial derivative is taken with respect to either variable. Thus, beginning with a function that is even in both x and y, and taking the first partial derivative with respect to x will produce a second function which is odd in x, and even in y. Similarly, taking the second partial derivative with respect to x of the initial function produces a third function which is even in x, and even in y. Since the stresses, σ_x and σ_y, for this particular problem are even in both x and y, it follows that the stress function is likewise even in x and y.

Examination of equation (23) reveals that the complementary solution is odd in both x and y if the coefficients a_i and b_i are different from zero. In order to resolve this conflict, it is required that these two coefficients be zero on both boundaries. That these coefficients are indeed zero will be shown for this particular problem, but a more general treatment would extend the result to other similar problems.

Effect of Solution Properties upon Integrals

Integral Operators

In that the contour C_i which is specified for each of the line integrals is any closed path enclosing an internal boundary, let the contour be a circle with its center at the origin. The normal derivative to the path, then, is the derivative in the radial direction, and the tangential derivative is the derivative with respect to the angle, θ, multiplied by the reciprocal of the radius, or

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$$

$$ds = r d\theta$$

and,

$$\frac{\partial}{\partial s} = \frac{1}{r}\frac{\partial}{\partial \theta}$$

The angle θ being measured from the positive x-axis, we have

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Since the radius is constant, it may be factored out of the integrals, and we have, for any function $\psi(r,\theta)$,

$$I_1\left\{\psi\right\}_i = \int_{c_i} \frac{\partial\psi}{\partial n}\,ds = r_i \int_0^{2\pi} \frac{\partial\psi}{\partial r}\,d\theta$$

$$I_2\left\{\psi\right\}_i = \int_{c_i}\left[y\,\frac{\partial\psi}{\partial n} - x\,\frac{\partial\psi}{\partial s}\right]ds = r_i \int_0^{2\pi}\left[(r_i \sin\theta)\,\frac{\partial\psi}{\partial r} - (\cos\theta)\,\frac{\partial\psi}{\partial}\right]d\theta$$

$$I_3\left\{\psi\right\}_i = \int_{c_i}\left[y\,\frac{\partial\psi}{\partial s} + x\,\frac{\partial\psi}{\partial n}\right]ds = r_i \int_0^{2\pi}\left[(\sin\theta)\,\frac{\partial\psi}{\partial} + (r_i \cos\theta)\,\frac{\partial\psi}{\partial r}\right]d\theta$$

$$I_4\left\{\psi\right\}_i = \int_{c_i} \psi\,ds = r_i \int_0^{2\pi} \psi\,d\theta$$

Odd and Even Integrands

Letting "I" denote any of the above integrals, with integrand $f(r,\theta)$, we have

$$I\left\{\psi\right\}_i = \int_0^{2\pi}\left[f(r,\theta)\right]d\theta = \int_0^{\pi/2}\left[f(r,\theta)\right]d\theta + \int_{\pi\,2}^{\pi}\left[f(r,\theta)\right]d\theta$$
$$+ \int_{\pi}^{3\frac{\pi}{2}}\left[f(r,\theta)\right]d\theta + \int_{\frac{3\pi}{2}}^{2\pi}\left[f(r,\theta)\right]d\theta$$

For the case of $f(r,\theta)$ being even in x and odd in y, we have

$$\int_0^{\frac{\pi}{2}}\left[f(r,\theta)\right]d\theta = \int_{\frac{\pi}{2}}^{\pi}\left[f(r,\theta)\right]d\theta = -\int_{\pi}^{\frac{3\pi}{2}}\left[f(r,\theta)\right]d\theta = -\int_{\frac{3\pi}{2}}^{2\pi}\left[f(r,\theta)\right]d\theta$$

or

$$I\left\{\psi\right\}_i = 0$$

For the case of $f(r,\theta)$ being odd in x and even in y, we have

$$\int_0^{\frac{\pi}{2}} \left[f(r,\theta)\right]d\theta = - \int_{\frac{\pi}{2}}^{\pi} \left[f(r,\theta)\right]d\theta = - \int_{\pi}^{\frac{3\pi}{2}} \left[f(r,\theta)\right]d\theta = \int_{\frac{3\pi}{2}}^{2\pi} \left[f(r,\theta)\right]d\theta$$

or,

$$I\left\{\psi\right\}_1 = 0$$

For the case of $f(r,\theta)$ being odd in x and odd in y, we have

$$\int_0^{\frac{\pi}{2}} \left[f(r,\theta)\right]d\theta = - \int_{\frac{\pi}{2}}^{\pi} \left[f(r,\theta)\right]d\theta = \int_{\pi}^{\frac{3\pi}{2}} \left[f(r,\theta)\right]d\theta = - \int_{\frac{3\pi}{2}}^{2\pi} \left[f(r,\theta)\right]d\theta$$

or,

$$I\left\{\psi\right\}_1 = 0$$

For the case of $f(r,\theta)$ being even in x and even in y, we have

$$\int_0^{\frac{\pi}{2}} \left[f(r,\theta)\right]d\theta = \int_{\frac{\pi}{2}}^{\pi} \left[f(r,\theta)\right]d\theta = \int_{\pi}^{\frac{3\pi}{2}} \left[f(r,\theta)\right]d\theta = \int_{\frac{3\pi}{2}}^{2\pi} \left[f(r,\theta)\right]d\theta$$

or

$$I\left\{\psi\right\} = 4 \int_0^{\frac{\pi}{2}} \left[f(r,\theta)\right]d\theta$$

The conclusion is reached that any of the above integrals which does not possess an integrand which is even in both x and y will have a resulting value of zero upon evaluation.

With respect to x and y, the Laplacian of ϕ has the same odd and even properties as ϕ, because the Laplacian is composed of second derivatives with respect to either variable. Furthermore, $\cos\theta$ is odd in x, even in y, and $\sin\theta$ is even in x, and odd in y. Remembering that the product of two odd functions, or two even functions yields an even function, and that the product of an odd function and an even function is an odd function, the integrands shall be investigated. Also,

$$\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$$

$\underline{I_1\{\psi\}_i}$

If the convention $(+,-)$, is used to denote a function that is even in x and odd in y, etc., then we have

$$\cos\theta = (-,+)$$

$$\sin\theta = (+,-)$$

and we may represent $\frac{\partial\psi}{\partial r}$ by

$$\frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial x}(-,+) + \frac{\partial\psi}{\partial y}(+,-)$$

For $\frac{\partial\psi}{\partial r}$ to be even in both x and y, it is obvious that we must have

$$\frac{\partial\psi}{\partial x} = (-,+)$$

$$\frac{\partial\psi}{\partial y} = (+,-)$$

or,

$$\psi = (+,+)$$

Thus, for $I_1 \left\{\psi\right\}_i$ to be non-zero, its argument, ψ, must be even in both x and y.

$\underline{I_2 \left\{\psi\right\}_i}$

From the integrand of this integral, we obtain

$$\left[(r_i \sin \theta) \frac{\partial\psi}{\partial r} - (\cos \theta) \frac{\partial\psi}{\partial\theta}\right] = \left[(2r_i \sin \theta \cos \theta) \frac{\partial\psi}{\partial x} + (r_i \sin^2 \theta - r_i \cos^2 \theta) \frac{\partial\psi}{\partial y}\right]$$

or

$$\text{integrand} = (-,-) \frac{\partial\psi}{\partial x} + (+,+) \frac{\partial\psi}{\partial y}$$

For this integrand to be even in x and y, it is necessary that ψ be even in x, and odd in y. All other combinations will yield a result of zero.

$\underline{I_3 \left\{\psi\right\}_i}$

From the integrand of this integral, we obtain

$$\left[(\sin \theta) \frac{\partial\psi}{\partial\theta} + (r_i \cos \theta) \frac{\partial\psi}{\partial r}\right] = \left[(-r_i \sin^2 \theta + r_i \cos^2 \theta) \frac{\partial\psi}{\partial x} + (2r_i \sin \theta \cos \theta) \frac{\partial\psi}{\partial y}\right]$$

or

$$\text{integrand} = (+,+) \frac{\partial\psi}{\partial x} + (-,-) \frac{\partial\psi}{\partial y}$$

For this integrand to be even in x and y, it is necessary that ψ be odd in x, and even in y. All other combinations will yield a result of zero.

$I_4\left\{\psi\right\}_i$

By inspection, it is seen that ψ must be even in both x and y for this integral to have a non-zero result.

Reduction of Composite Solution

Applying the results of the previous section, it will be possible to inspect the argument variable of each integral, and immediately tell whether the integral is required to possess a result of zero. If the integral does not possess a result of zero by inspection, it may still possibly turn out to be equal to zero upon evaluation. In the presentation to follow, "non-zero" will denote the value of an integral that is not zero by inspection.

In Table 1, the argument variables, their properties in x and y, and their consequent effect upon the line integrals are all tabulated. Referring to Table 1 and equations (34) through (36), we have

$$c_1 I_1\left\{\nabla^2\phi_{c_1}\right\}_i = -I_1\left\{\nabla^2\phi_d\right\}_1 - (1-\nu)\, I_1\left\{V\right\} \tag{37}$$

$$b_1 I_2\left\{\nabla^2\phi_{b_1}\right\}_1 = 0$$

$$a_1 I_3\left\{\nabla^2\phi_{a_1}\right\}_1 = 0$$

From this, it may be concluded that

$$a_1 = b_1 = 0$$

Table 1

Argument Variable:	Properties in x	Properties in y	I_1 1	I_2 1	I_3 1	I_4 1
$\nabla^2\phi_{a_1}$	odd	even	0	0	NZ	-
$\nabla^2\phi_{b_1}$	even	odd	0	NZ	0	-
$\nabla^2\phi_{c_1}$	even	even	NZ	0	0	-
$\nabla^2\phi_d$	even	even	NZ	0	0	-
V	even	even	NZ	0	0	-
Vm	even	odd	-	-	-	0
$V\ell$	odd	even	-	-	-	0

NZ: Non-zero

- : Not applicable

$$\phi = c_1 \phi_{c_1} + \phi_d \tag{38}$$

The boundary conditions are then reduced to

$$\left.\phi^*\right)_1 = C_1 \tag{39}$$

$$\left.\phi^*\right)_2 = 0 \tag{40}$$

$$\left.\phi_d\right)_i = \frac{\rho\omega^2}{4} R_i^4 \tag{41}$$

$$\left.\frac{\partial\phi^*}{\partial x}\right)_i = \left.\frac{\partial\phi^*}{\partial y}\right)_i = 0 \quad (i = 1,2) \tag{42}$$

$$\left.\frac{\partial\phi_d}{\partial x}\right)_i = \frac{\rho\omega^2}{2} R_i^2 x \tag{43}$$

$$\left.\frac{\partial\phi_d}{\partial y}\right)_i = \frac{\rho\omega^2}{2} R_i^2 y \tag{44}$$

The composite solution has been reduced from a total of four stress functions to the above two. Assuming that ϕ_{c_1} and ϕ_d may be found, the constant c_1 may be computed from equation (37), and the stress function ϕ from equation (38). As mentioned at the end of Chapter III, the stress distribution may then be obtained from equations (9) through (11).

CHAPTER V

METHOD OF SOLUTION

Finite Difference Formulation

In the preceding chapters, it was assumed that a method existed for evaluating the stress function, ϕ, and indeed, several such methods exist. Basically, the problem consists of the solution of the non-homogeneous biharmonic equation (21) subject to the boundary conditions (15) through (17), and (39) through (41). The resulting solution will be a unique stress function which may be substituted into equations (9), (10), and (11) to obtain expressions for the stresses.

A general solution to the biharmonic equation does exist, but its complexity makes prohibitive the task of evaluating the many constants it contains. This is due to the fact that these constants are evaluated by applying the boundary conditions to the general solution, a somewhat formidable endeavor, at best. The usual procedure for obtaining a closed-form solution is to obtain a number of elemental solutions which satisfy the biharmonic equation, and then superimpose these solutions to obtain an "exact" solution. The solution so obtained is exact insofar as the biharmonic equation is concerned, but not so, for the boundary conditions. Theoretically, the boundary conditions could also be satisfied exactly if enough of these elemental solutions were used, but the resulting solution becomes extremely unwieldy. With regard to these advanced exact methods, Manson[8] states that

"Although the methods are exact in that each of the elemental solutions satisfies the biharmonic equation, and the exact solution can presumably be obtained by the superposition of a sufficient number of these solutions, they are approximate when practical considerations are injected into the problem of satisfying the boundary conditions. The calculations become cumbersome, particularly in the methods involving the application of roots of transcendental equations and complicated combinations of complex numbers.

In practice, therefore, the results obtained become no more accurate, and in many cases, much less accurate, than approximate methods. One reason for this is that, while the form of solution is such that the differential equation is always satisfied, the degree to which the boundary conditions are satisfied, depends markedly upon the number of terms used. Unless many terms are used, adherence to the boundary conditions may be very poor.

On the other hand, it is the boundary conditions, rather than the biharmonic equation, that have the most influential effect in defining the specific nature of a given elasticity problem. Hence, superior results may be obtained by methods in which the assumed solutions are such that the boundary conditions are exactly satisfied, while adherence to the differential equation at all points is compromised by limitations in the number of terms used."

Of the available approximate methods for solving the biharmonic equation, the method of finite differences appears to be the most promising. Recalling that the finite difference method involves solving for the argument variable of a differential equation at a predetermined number of isolated points, it must be realized that the accuracy of the solution may be progressively improved by increasing the number of points used. By this method, then, an investigator may easily obtain a somewhat crude approximation to the solution by using a few points, and later improve the accuracy to the desired degree by simply increasing the number of points in the grid.

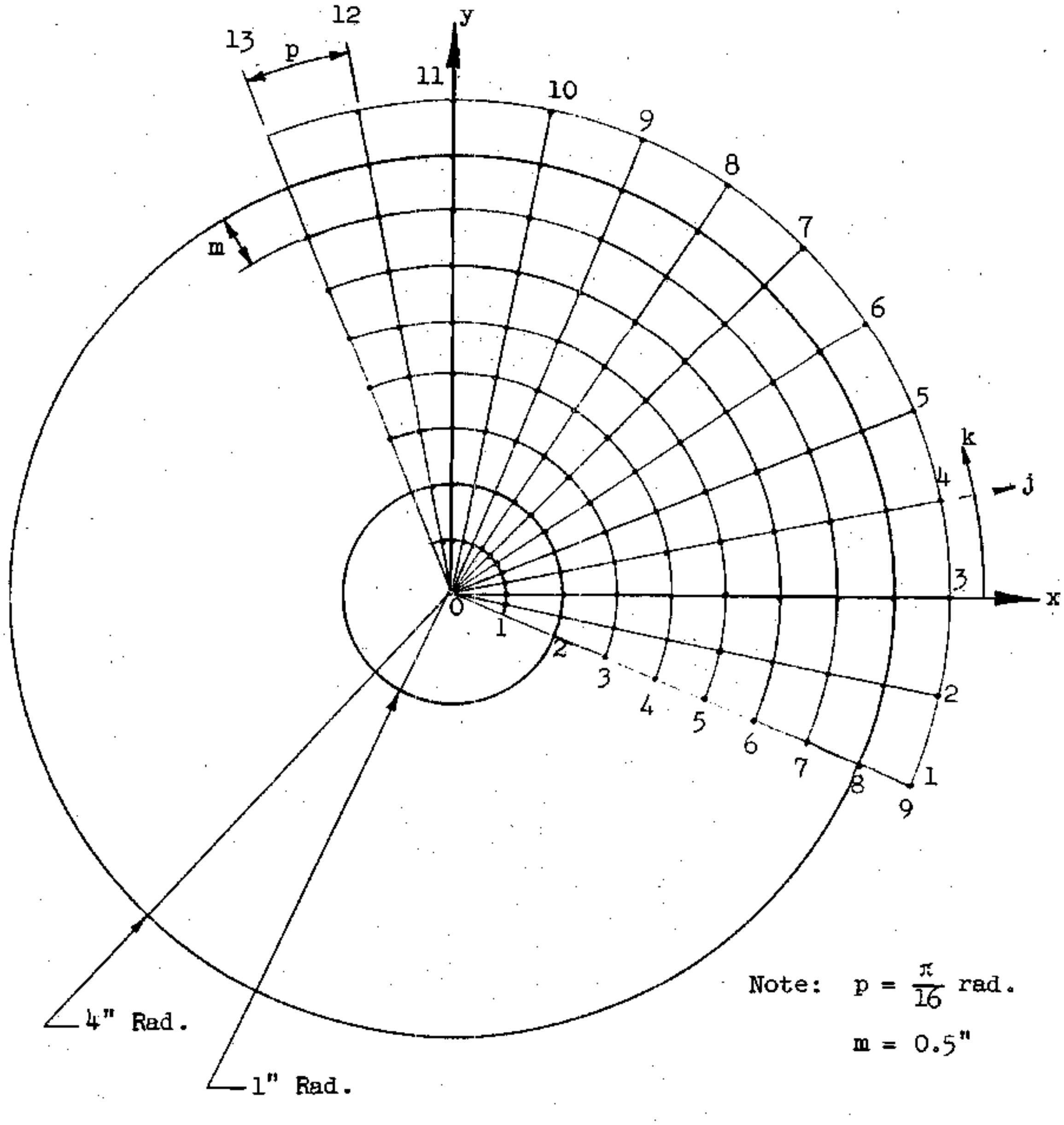
Relaxation techniques for solving systems of finite difference equations have seen broad application in all fields of engineering, and when programmed for high-speed digital computers, they provide a very practical method for obtaining an approximate solution to a problem such as this.

In that a method of solution was desired which would be practical from the standpoint of engineering design, it was decided to use the finite difference approximation to solve the problem. To further make the method as rapid and simple as possible, the relaxation procedure, programmed for the Burroughs 220 digital computer, was used to solve the resulting systems of equations.

Polar Grid

In most applications, a square grid, or network, of points is used, and the finite difference equation expressed in Cartesian coordinates. For bodies having rectangular boundaries, such an approach is logical, as the grid size may be adjusted to make each boundary coincide with a line of grid points. In any application where defined boundaries pass between grid points, such as curved boundaries and rectangular grids, approximations must be used to define certain of these boundary-adjacent points in terms of other points which are better defined. This is an undesirable situation at best, because it means that there are approximations within an already approximate method, and accuracy must surely suffer. The best approach, then, is to use a grid network of a type which best fits the boundaries of the body in question.

In that the disk of this problem possesses two concentric, circular boundaries, it was decided to use a polar grid of the type shown in Figure 2, and to express the biharmonic equation in terms of polar coordinates. The grid size was adjusted to fit the boundaries as well as



Note: $p = \frac{\pi}{16}$ rad.

$m = 0.5"$

Figure 2. Disk with Grid Network.

possible, consistent with the number of grid points deemed feasible for an initial investigation.

Biharmonic Equation

In order to express the biharmonic equation in polar finite difference form, the following finite difference expressions are needed. Where m is the grid size in the radial direction, and p is the grid size in the tangential direction, we have:

$$\Delta r = m; \quad \Delta\theta = p$$

and where j denotes the r coordinate, and k, the θ coordinate, we have

$$\left.\frac{\partial\phi}{\partial r}\right)_{j,k} = \frac{1}{2m} \left(\phi_{j+1,k} - \phi_{j-1,k}\right) \tag{45}$$

$$\left.\frac{\partial^2\phi}{\partial r}\right)_{j,k} = \frac{1}{m^2} \left(\phi_{j+1,k} - 2\phi_{j,k} + \phi_{j-1,k}\right) \tag{46}$$

$$\left.\frac{\partial^3\phi}{\partial r^3}\right)_{j,k} = \frac{1}{2m^3} \left(\phi_{j+2,k} - 2\phi_{j+1,k} + 2\phi_{j-1,k} - \phi_{j-2,k}\right) \tag{47}$$

$$\left.\frac{\partial^4\phi}{\partial r^4}\right)_{j,k} = \frac{1}{m^4} \left(\phi_{j+2,k} - 4\phi_{j+1,k} + 6\phi_{j,k} - 4\phi_{j-1,k} + \phi_{j-2,k}\right) \tag{48}$$

$$\left.\frac{\partial^2\phi}{\partial\theta^2}\right)_{j,k} = \frac{1}{p^2} \left(\phi_{j,k+1} - 2\phi_{j,k} + \phi_{j,k-1}\right) \tag{49}$$

$$\left.\frac{\partial^4\phi}{\partial r^4}\right)_{j,k} = \frac{1}{p^4} \left(\phi_{j,k+2} - 4\phi_{j,k+1} + 6\phi_{j,k} - 4\phi_{j,k-1} + \phi_{j,k-2}\right) \tag{50}$$

$$\left.\frac{\partial^3\phi}{\partial r\partial\theta^2}\right)_{j,k} = \frac{1}{mp^2}\left(\phi_{j+1,k+1} - \phi_{j-1,k+1} - 2\phi_{j+1,k} + 2\phi_{j-1,k}\right. \quad (51)$$

$$\left. + \phi_{j+1,k-1} - \phi_{j-1,k-1}\right)$$

$$\left.\frac{\partial^4\phi}{\partial r^2\partial\theta^2}\right)_{j,k} = \frac{1}{m^2p^2}\left(\phi_{j+1,k+1} - 2\phi_{j,k+1} + \phi_{j-1,k+1} - 2\phi_{j+1,k}\right. \quad (52)$$

$$\left. + 4\phi_{j,k} - 2\phi_{j-1,k} + \phi_{j+1,k-1} - 2\phi_{j,k-1} + \phi_{j-1,k-1}\right)$$

Referring to the biharmonic equation (21), it is seen that certain of the above derivatives possess coefficients involving the radius, r. In the finite difference formulation, the radius used will be that to the central point (j,k). Applying the proper coefficients, and combining terms, we obtain, for the right-hand side of (21)

$$\left.\nabla^4\phi\right)_{j,k} = a_1(r)\ \phi_{j+2,k} + a_2(r)\ \phi_{j+1,k} + a_3(r)\ \phi_{j,k} + a_4(r)\phi_{j-1,k} \quad (53)$$

$$+ a_5(r)\phi_{j-2,k} + a_6(r)\ (\phi_{j,k+2} + \phi_{j,k-2}) + a_7(r)(\phi_{j+1,k+1}$$

$$+ \phi_{j+1,k-1}) + a_8(r)(\phi_{j,k+1} + \phi_{j,k-1}) + a_9(r)(\phi_{j-1,k+1}$$

$$+ \phi_{j-1,k-1})$$

where

$$a_1(r) = \left(\frac{1}{m^4} + \frac{1}{rm^3} \right)$$

$$a_2(r) = \left(- \frac{4}{m^4} - \frac{2}{rm^3} - \frac{1}{r^2m^2} + \frac{1}{r^3mp^2} + \frac{4}{r^3mp^2} - \frac{4}{r^2m^2p^2} \right)$$

$$a_3(r) = \left(\frac{6}{m^4} + \frac{2}{r^2m^2} + \frac{8}{r^2m^2p^2} - \frac{8}{r^4p^2} + \frac{6}{r^4p^4} \right)$$

$$a_4(r) = \left(-\frac{4}{m^4} + \frac{2}{rm^3} - \frac{1}{r^2m^2} - \frac{1}{2r^3m} - \frac{4}{r^3mp^2} - \frac{4}{r^2m^2p^2}\right)$$

$$a_5(r) = \left(\frac{1}{m^4} - \frac{1}{rm^3}\right)$$

$$a_6(r) = \frac{1}{r^4p^4}$$

$$a_7(r) = \left(-\frac{2}{r^3mp^2} + \frac{2}{r^2m^2p^2}\right)$$

$$a_8(r) = \left(-\frac{4}{r^2m^2p^2} + \frac{4}{r^4p^2} - \frac{4}{r^4p^4}\right)$$

$$a_9(r) = \left(\frac{2}{r^3mp^2} + \frac{2}{r^2m^2p^2}\right)$$

Thus, the finite difference expression of the biharmonic equation at a central point (j,k) involves the value of ϕ at the central point plus the values at twelve other surrounding points. This array of points is often referred to as the "biharmonic star,"[9] and is illustrated in Figure 3 with the coefficients of the various points superimposed.[8]

The boundary conditions at the two boundaries, equations (39) through (44), must all be expressed in terms of finite differences and/or the coordinates, depending on whether derivatives are involved. Use of values of the derivatives on the boundaries allows values to be affixed to the fictitious points which do not lie on the disk. These points must be used in order to obtain a solution over the complete grid. In the same vein, the boundary conditions set the values of the stress function at points which lie on the boundaries.

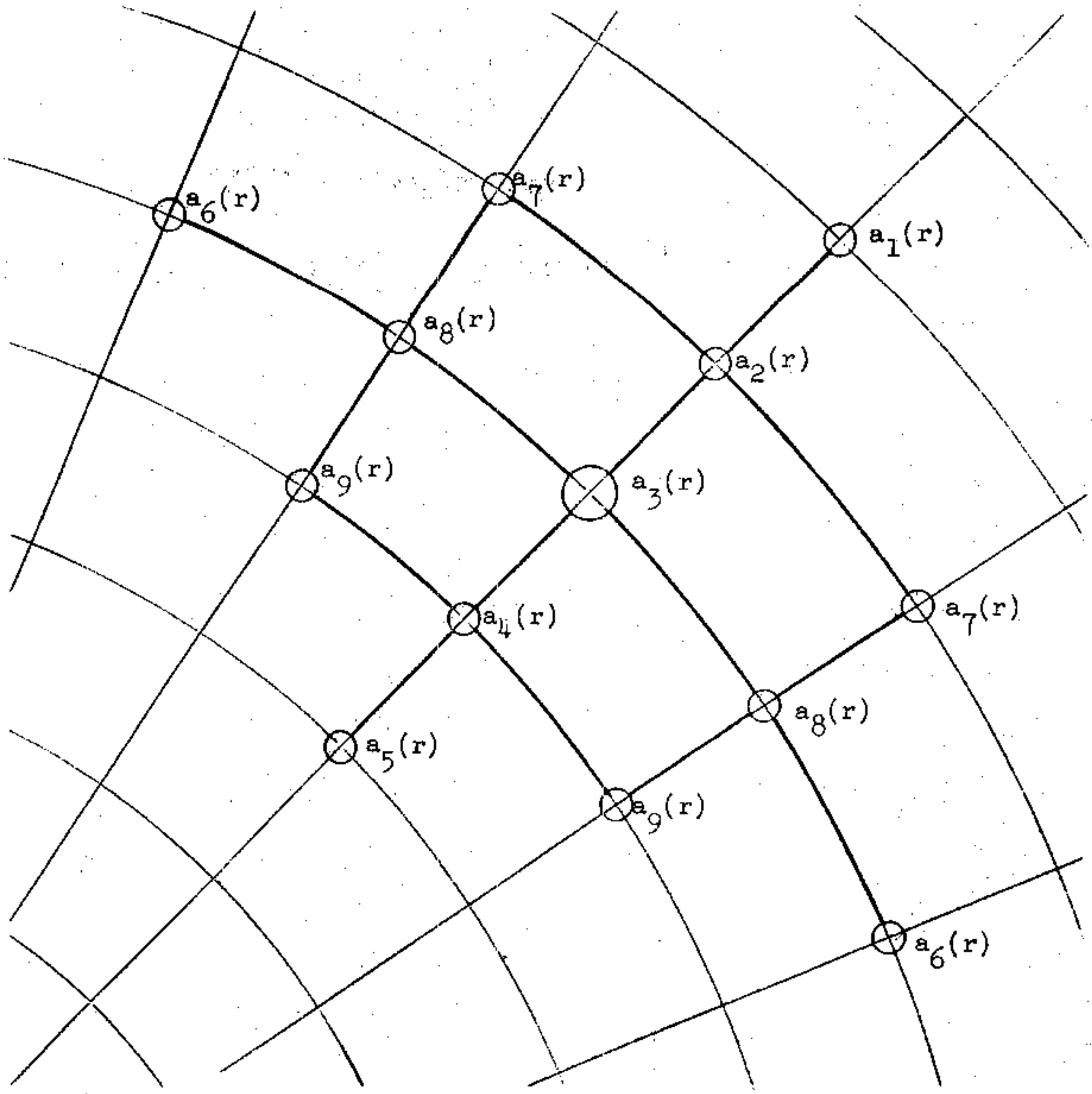


Figure 3. Biharmonic Star.

Solution Procedure

The composite solution (38) was employed, therefore, the biharmonic equation had to be solved over the grid one time for each of the two stress functions ϕ_{c_1}, and ϕ_d. For the stress function ϕ_{c_1}, the homogeneous biharmonic equation was solved, and for ϕ_d, the nonhomogeneous biharmonic was similarly solved. In each case, a relaxation procedure was used such that the value of the stress function at the central point was recomputed to make the associated residual, of error, zero before moving to the next grid point. The method of sweeping over the grid was accomplished as follows: (a) Starting at the first arc of grid points lying between the inner and outer boundaries, at θ equal to zero, advance in the positive tangential direction to the edge of the quadrant. (b) Increase radius one increment, and repeat until last arc inside of outer boundary is reached. (c) Return to starting point. Steps (a) through (c) represent one sweep of the entire grid. Iteration and relaxation was accomplished by repeated sweeps. After sufficient iteration had been carried out to reduce the residuals at all points to negligible magnitude, the stress function values at the various points were assumed to be correct. In each case, 500 sweeps over the grid reduced the residuals such that there was no change in the eighth significant digit for succeeding iterations, thus, further iterations were deemed unnecessary.

After the 500 sweeps had been executed, finite differences were again employed to compute the Laplacian of the stress function at all relevant points. These values of the Laplacian were later used in evaluating the line integrals of Michell's conditions.

Evaluation of the Constant, c_1

In order to evaluate the constant, c_1, it was first necessary to evaluate the line integrals appearing in equation (37). The integral of the potential function was easily found in closed form, but numerical integration, using Simpson's Rule, was employed on the two other integrals involving the Laplacian of a stress function. The values of the two stress functions and their Laplacians were all fed into a final computer program as input data. This final program performed the numerical integrations, computed c_1, the final stress function ϕ, the approximate stress distribution using preceding results, and finally, the actual stress distribution using the known solution to the problem. The value of the constant c_1 was found to be 502.9437. The values of ϕ_{c_1}, ϕ_d, and the final stress function ϕ are shown in Figure 4. These values were found to be the same along each radial line of grid points.

Stress Distribution

The final stress distribution resulting from the above computations is shown in Figure 5. Also plotted is the exact stress distribution which was computed from the equations

$$\sigma_r = \frac{3+\nu}{8} \rho\omega^2 \left(b^2 + a^2 - \frac{a^2 b^2}{r^2} - r^2\right) \tag{54}$$

$$\sigma_\theta = \frac{3+\nu}{8} \rho\omega^2 \left(b^2 + a^2 + \frac{a^2 b^2}{r^2} - \frac{1 + 3\nu}{3 + \nu} r^2\right) \tag{55}$$

where a and b denote the inner and outer radii respectively. A derivation of these equations is found in Reference 1, p. 70.

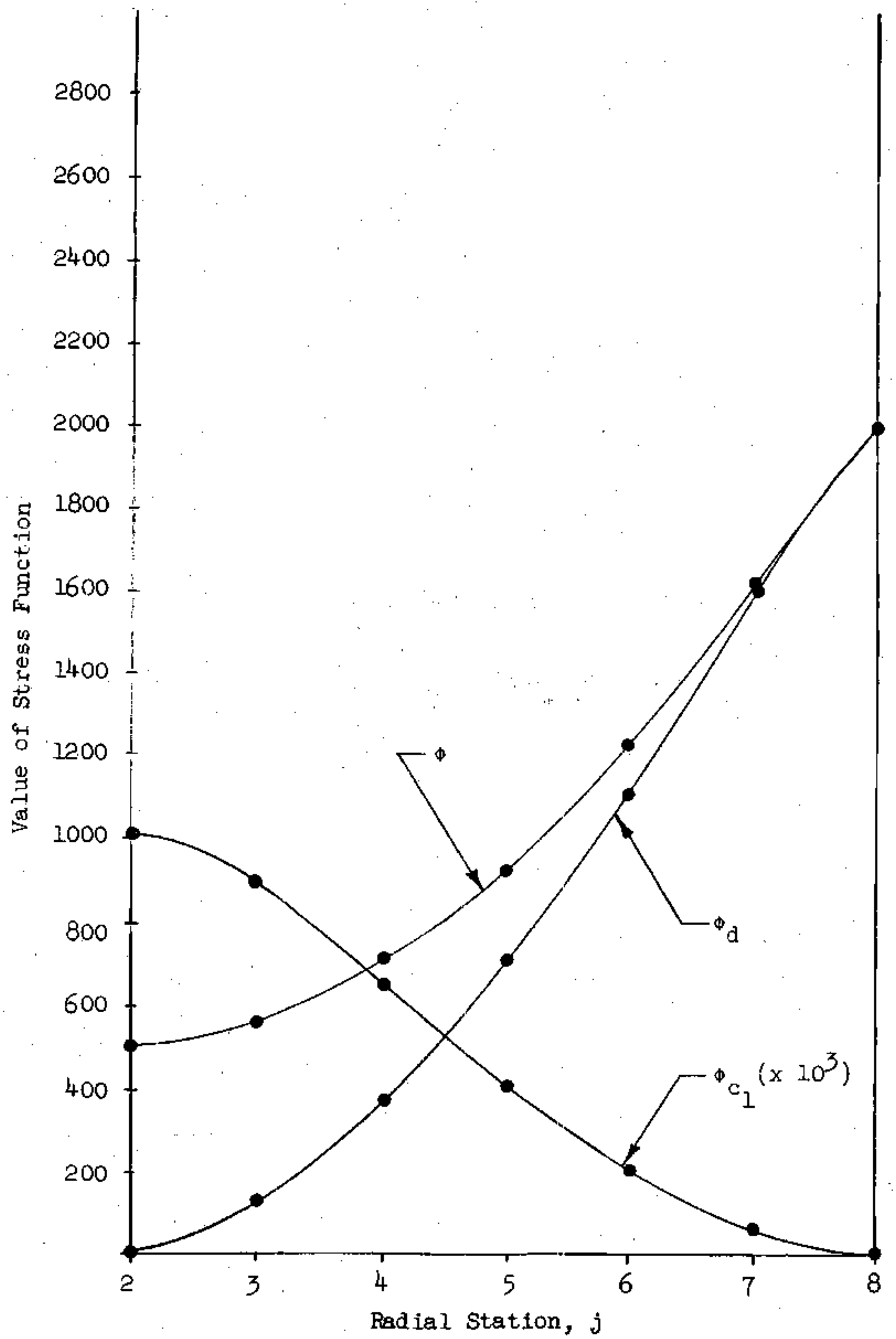


Figure 4. Stress Functions.

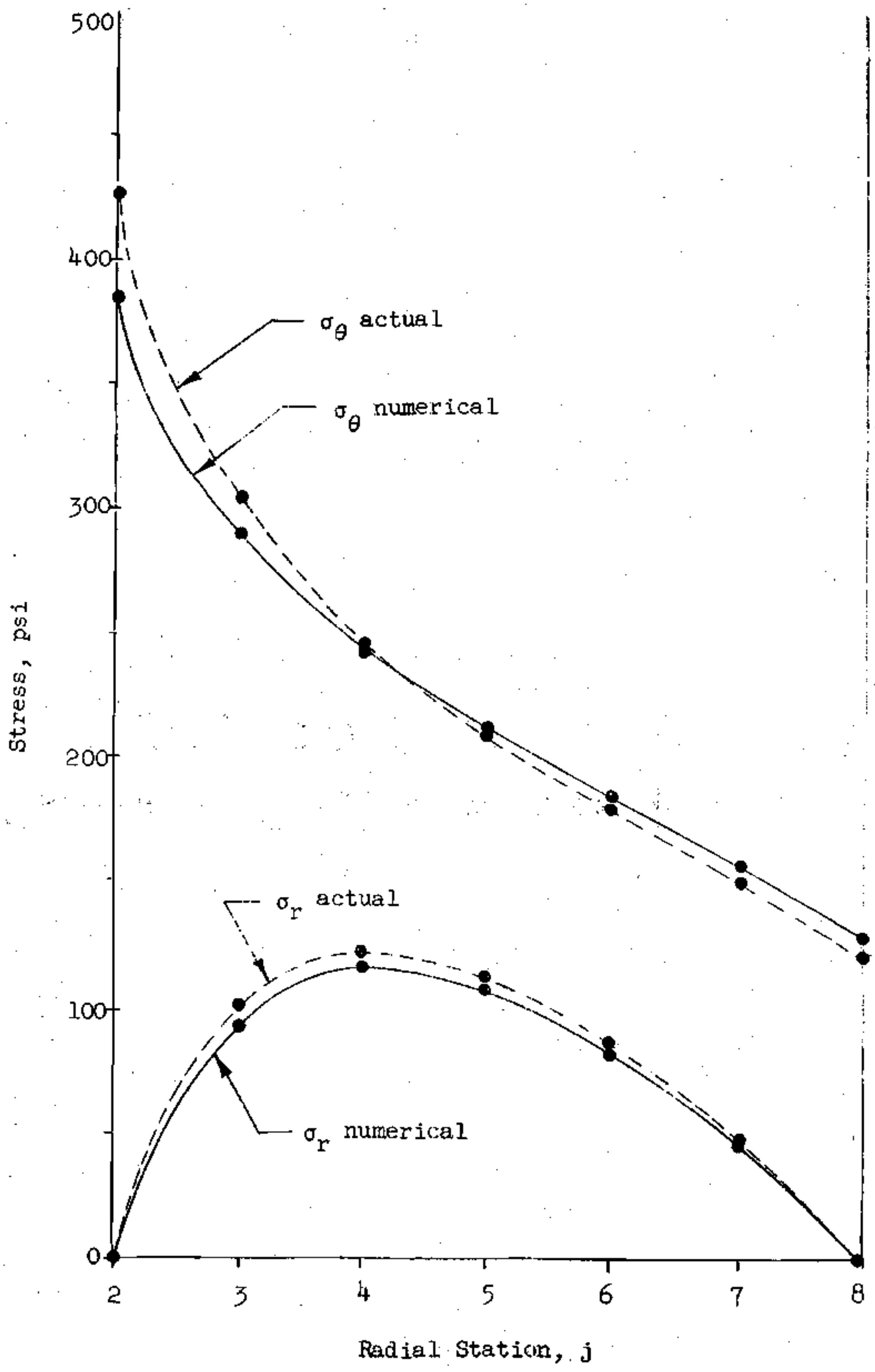


Figure 5. Stress Distribution in Disk.

CHAPTER VI

DISCUSSION OF RESULTS

Conclusions

Inspection of Figure 5 reveals that the numerical solution is in fairly close agreement with the exact solution for the values of the stresses. The error in the tangential stress at the boundary of the central hole is 11.3 per cent, and it should be noted that this is the point of maximum stress. Due to the fact that a small number of grid points were used, and since the stress gradient at the inner boundary is quite high, it is not unreasonable to expect errors of this order of magnitude. In that the rotational speed of disk in this investigation was only 2000 RPM, and that the stress varies as the square of the speed (equations (54) and (55)), it can be seen that much higher rotational speeds will produce severe stress gradients at the inner boundary. Large grid spacings used in the presence of high stress gradients are bound to produce inaccurate or misleading results. In order to maintain a given standard of accuracy, it is mandatory that the investigator decrease the grid spacing in accordance with any increases in speed. In the event that many grid points are to be used, it may be advisable to first apply a rather coarse grid network to obtain approximate values of the stress functions. Approximate initial values at all points on the finer grid may be obtained by interpolation of the initial results, and thereby reduce the number of necessary iterations, and hence, computation time, considerably.

At the outset, it was realized that use of a polar grid meant an increasing size of grid blocks in the radial direction, and that accuracy would probably decrease noticeably in the radial direction. The results obtained indicate that values of the stress gradients have a more noticeable effect upon the accuracy than does the radial change in grid size, as the error in the tangential stress at the inner boundary is 11.32 per cent, and the error at the outer boundary is 10.95 per cent.

The postulate of a symmetrical stress distribution for this particular problem was made in an earlier chapter, and many subsequent manipulations were based on this condition. Examination of the data in Appendix B will reveal that the results obtained through the presented method of solution agree with the postulate. Such a test of symmetry is extremely valuable when examining output data from a relaxation procedure to determine whether sufficient iterations have been executed.

The computer programs used to obtain the values of the stress functions are presented in Appendix C, and are followed by the final program which computed the stress distribution using as input data, the value computed in the first two programs. The values of ϕ_{c_1} in Appendix B have been multiplied by 10^4.

In light of the results obtained in solving this one-dimensional problem as if it were, in actuality, two-dimensional, it is concluded that numerical techniques applied to current elastic theory offer feasible and accurate solutions to otherwise intractable two-dimensional rotating disk problems.

Recommendations

It is recommended that future investigations be conducted to determine the practical level of accuracy attainable in using the method herein presented for the solution of actual two-dimensional rotating disk problems. It is further recommended that consideration be given to programming a general computer procedure to solve a wide variety of, if not all, rotating disk problems. Other variations of the rotating disk which are of interest are those with multiple internal boundaries, steady-state or transient heat flux, variable disk thickness, external loadings, and angular acceleration.

APPENDIX A

Beginning with the strain-deformation relations, we have

$$\epsilon_r = \frac{\partial u}{\partial r} \tag{a}$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \tag{b}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{c}$$

Now, from (c), we have

$$\frac{\partial \gamma_{r_\theta}}{\partial \theta} = \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 v}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and taking the partial derivative with respect to r, we obtain

$$\frac{\partial^2 \gamma_{r_\theta}}{\partial r \partial \theta} = -\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial^3 u}{\partial r \partial \theta^2} + \frac{\partial^3 v}{\partial r^2 \partial \theta} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta}$$

From these last two expressions, we obtain

$$\frac{\partial^2 \gamma_{r_\theta}}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial \gamma_{r_\theta}}{\partial \theta} = \frac{1}{r}\frac{\partial^3 u}{\partial r \partial \theta^2} + \frac{\partial^3 v}{\partial r^2 \partial \theta}$$

but

$$\frac{\partial^3 u}{\partial r \partial \theta^2} = \frac{\partial^2 \epsilon_r}{\partial \theta^2}$$

Thus,

$$\frac{\partial^2 \gamma_{r_\theta}}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial \gamma_{r_\theta}}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \epsilon_r}{\partial \theta^2} = \frac{\partial^3 v}{\partial r^2 \partial \theta} \qquad (d)$$

Now, an expression in terms of the strains must be found for $\frac{\partial^3 v}{\partial r^2 \partial \theta}$

We have

$$\frac{\partial \epsilon_\theta}{\partial r} = - \frac{1}{r^2} u + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} \frac{\partial v}{\partial \theta} + \frac{1}{r} \frac{\partial^2 v}{\partial r \partial \theta} ,$$

which leads to

$$\frac{\partial \epsilon_\theta}{\partial r} + \frac{1}{r} \epsilon_\theta = \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial^2 v}{\partial r \partial \theta}$$

but

$$\frac{\partial u}{\partial r} = \epsilon_r$$

Thus, we have

$$r \frac{\partial \epsilon_\theta}{\partial r} + \epsilon_\theta - \epsilon_r = \frac{\partial^2 v}{\partial r \partial \theta}$$

Taking another partial derivative with respect to r, we have

$$r \frac{\partial^2 \epsilon_\theta}{\partial r^2} + 2 \frac{\partial \epsilon_\theta}{\partial r} - \frac{\partial \epsilon_r}{\partial r} = \frac{\partial^3 v}{\partial r^2 \partial \theta}$$

Substituting this expression for the right-hand side of (d), we obtain

$$\frac{\partial^2 \gamma_{r_\theta}}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial \gamma_{r_\theta}}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \epsilon_r}{\partial \theta^2} - r\frac{\partial^2 \epsilon_\theta}{\partial r^2} - 2\frac{\partial \epsilon_\theta}{\partial r} + \frac{\partial \epsilon_r}{\partial r} = 0$$

Factoring out the strains, we have

$$\left(\frac{\partial^2}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\right)\gamma_{r_\theta} + \left(\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\right)\epsilon_r - \left(r\frac{\partial^2}{\partial r^2} + 2\frac{\partial}{\partial r}\right)\epsilon_\theta = 0$$

which is the mathematical relationship between the strains, or the compatibility equation.

APPENDIX B

On the following pages, the values of ϕ_{c_1}, ϕ_d, and ϕ are listed as are the values of the radial stress, tangential stress, and shear stress for both the numerical and exact solutions.

```
2COMMENT D.E. TATE, VALUES OF AIRY STRESS FUNCTION-DISK WITH CENTRAL
2        HOLE                                                          $
2COMMENT PHC1 VALUES                                                   $
2ARRAY   PH(9,13),A1(7),A2(7),A3(7),A4(7),A5(7),A6(7),A7(7),A8(7),A9(7)
2        ,DSQ(9,13)                                                    $
2INTEGER J,K                                                           $
2JPOT..  READ($$DATA)                                                  $
2        FOR J=(3,1,7)                                                 $
2BEGIN   B1=1/(M*4)                                                    $
2        B2=1/((J)(M*4))                                               $
2        B3=1/((J*2)(M*4))                                             $
2        B4=1/(2(J*3)(M*4))                                            $
2        B5=2/((J*3)(M*4)(P*2))                                        $
2        B6=2/((J*2)(M*4)(P*2))                                        $
2        B7=4/((J*4)(M*4)(P*2))                                        $
2        B8=1/((J*4)(M*4)(P*4))                                        $
2        A1(J)=B1+B2                                                   $
2        A2(J)=-4.B1-2.B2-B3+B4+2.B5-2.B6                              $
2        A3(J)=6.B1+2.B3+4.B6-2.B7+6.B8                                $
2        A4(J)=-4.B1+2.B2-B3-B4-2.B5-2.B6                              $
2        A5(J)=B1-B2                                                   $
2        A6(J)=B8                                                      $
2        A7(J)=-B5+B6                                                  $
2        A8(J)=-2.B6+B7-4.B8                                           $
2        A9(J)=B5+B6                                      END          $
2BOUND1..PH(2,3)=1.0                                                   $
2        PH(2,4)=1.0                                                   $
2        PH(2,5)=1.0                                                   $
2        PH(2,6)=1.0                                                   $
2        PH(2,7)=1.0                                                   $
2        PH(2,8)=1.0                                                   $
2        PH(2,9)=1.0                                                   $
2        PH(2,10)=1.0                                                  $
2        PH(2,11)=1.0                                                  $
2BOUND2..PH(8,3)=0.0                                                   $
2        PH(8,4)=0.0                                                   $
```

```
2          PH(8,5)=0.0                                                    $
2          PH(8,6)=0.0                                                    $
2          PH(8,7)=0.0                                                    $
2          PH(8,8)=0.0                                                    $
2          PH(8,9)=0.0                                                    $
2          PH(8,10)=0.0                                                   $
2          PH(8,11)=0.0                                                   $
2SYMX..    PH(2,2)=+PH(2,4)                                               $
2          PH(3,2)=+PH(3,4)                                               $
2          PH(4,2)=+PH(4,4)                                               $
2          PH(5,2)=+PH(5,4)                                               $
2          PH(6,2)=+PH(6,4)                                               $
2          PH(7,2)=+PH(7,4)                                               $
2          PH(8,2)=+PH(8,4)                                               $
2          PH(3,1)=+PH(3,5)                                               $
2          PH(4,1)=+PH(4,5)                                               $
2          PH(5,1)=+PH(5,5)                                               $
2          PH(6,1)=+PH(6,5)                                               $
2          PH(7,1)=+PH(7,5)                                               $
2SYMY..    PH(2,12)=+PH(2,10)                                             $
2          PH(3,12)=+PH(3,10)                                             $
2          PH(4,12)=+PH(4,10)                                             $
2          PH(5,12)=+PH(5,10)                                             $
2          PH(6,12)=+PH(6,10)                                             $
2          PH(7,12)=+PH(7,10)                                             $
2          PH(8,12)=+PH(8,10)                                             $
2          PH(3,13)=+PH(3,9)                                              $
2          PH(4,13)=+PH(4,9)                                              $
2          PH(5,13)=+PH(5,9)                                              $
2          PH(6,13)=+PH(6,9)                                              $
2          PH(7,13)=+PH(7,9)                                              $
2OUT1..    PH(1,3)=PH(3,3)                                                $
2          PH(1,4)=PH(3,4)                                                $
2          PH(1,5)=PH(3,5)                                                $
2          PH(1,6)=PH(3,6)                                                $
2          PH(1,7)=PH(3,7)                                                $
```

```
2           PH(1,8)=PH(3,8)                                                    $
2           PH(1,9)=PH(3,9)                                                    $
2           PH(1,10)=PH(3,10)                                                  $
2           PH(1,11)=PH(3,11)                                                  $
2OUT2..     PH(9,3)=PH(7,3)                                                    $
2           PH(9,4)=PH(7,4)                                                    $
2           PH(9,5)=PH(7,5)                                                    $
2           PH(9,6)=PH(7,6)                                                    $
2           PH(9,7)=PH(7,7)                                                    $
2           PH(9,8)=PH(7,8)                                                    $
2           PH(9,9)=PH(7,9)                                                    $
2           PH(9,10)=PH(7,10)                                                  $
2           PH(9,11)=PH(7,11)                                                  $
2           C=C+1                                                              $
2           FOR J=(3,1,7)                                                      $
2           FOR K=(3,1,11)                                                     $
2           PH(J,K)=-(1/A3(J))(A1(J).PH(J+2,K)+A2(J).PH(J+1,K)
2           +A4(J).PH(J-1,K)+A5(J).PH(J-2,K)+A6(J).(PH(J,K+2)+PH(J,K-2))
2           +A7(J)(PH(J+1,K+1)+PH(J+1,K-1))+A8(J)(PH(J,K+1)+PH(J,K-1))
2           +A9(J)(PH(J-1,K+1)+PH(J-1,K-1)))                                   $
2           IF C LEQ 500                                                       $
2           GO TO SYMX                                                         $
2           FOR J=(2,1,8)                                                      $
2           FOR K=(3,1,11)                                                     $
2           DSQ(J,K)=((1/(M*2))+(1/(2.J(M*2))))PH(J+1,K)-((2/(M*2))
2           +(2/((J*2)(M*2)(P*2))))PH(J,K)+((1/(M*2))-(1/(2.J(M*2))))PH(J
2           -1,K)+(1/((J*2)(M*2)(P*2)))(PH(J,K+1)+PH(J,K-1))                   $
2           WRITE($$ANS,FMT)                                                   $
2INPUT      DATA(M,P,C,FOR J=(2,1,8)$ FOR K=(3,1,11)$ PH(J,K))                 $
2OUTPUT     ANS(FOR J=(2,1,8) $ FOR K=(3,1,11) $ PH(J,K),FOR J=(2,1,8) $
2           FOR K=(3,1,11) $ DSQ(J,K))                                         $
2FORMAT     FMT(*5*,B2,3F15.8,W0)                                              $
2           FINISH                                                             $
```

```
2COMMENT D.E. TATE, VALUES OF AIRY STRESS FUNCTION-DISK WITH CENTRAL
2         HOLE                                                          $
2COMMENT PHD  VALUES                                                    $
2ARRAY    PH(9,13),A1(7),A2(7),A3(7),A4(7),A5(7),A6(7),A7(7),A8(7),A9(7)
2         ,DSQ(9,13),TH(11)                                             $
2INTEGER  J,K                                                           $
2JPOT..   READ($$DATA)                                                  $
2         FOR J=(3,1,7)                                                 $
2BEGIN    B1=1/(M*4)                                                    $
2         B2=1/((J)(M*4))                                               $
2         B3=1/((J*2)(M*4))                                             $
2         B4=1/(2(J*3)(M*4))                                            $
2         B5=2/((J*3)(M*4)(P*2))                                        $
2         B6=2/((J*2)(M*4)(P*2))                                        $
2         B7=4/((J*4)(M*4)(P*2))                                        $
2         B8=1/((J*4)(M*4)(P*4))                                        $
2         A1(J)=B1+B2                                                   $
2         A2(J)=-4.B1-2.B2-B3+B4+2.B5-2.B6                              $
2         A3(J)=6.B1+2.B3+4.B6-2.B7+6.B8                                $
2         A4(J)=-4.B1+2.B2-B3-B4-2.B5-2.B6                              $
2         A5(J)=B1-B2                                                   $
2         A6(J)=B8                                                      $
2         A7(J)=-B5+B6                                                  $
2         A8(J)=-2.B6+B7-4.B8                                           $
2         A9(J)=B5+B6                                   END             $
2         RHO=0.000738                                                  $
2         OMEGA=(2000.0)(8/15.0)(0.1963495)                             $
2         NU=0.30                                                       $
2         RO=(RHO)(OMEGA*2)                                             $
2         RA=1.0                                                        $
2         RB=4.0                                                        $
2         FOR K=(3,1,11)                                                $
2BEGIN
2BOUND1..   PH(2,K)=(RO/4.0)(RA*4)                                      $
2BOUND2..   PH(8,K)=(RO/4.0)(RB*4)                      END             $
2LOOP..   FOR K=(3,1,11)                                                $
```

```
2BEGIN
2OUT1..   PH(1,K)=PH(3,K)-RO.M(RA*3)                                  $
2OUT2..   PH(9,K)=PH(7,K)+RO.M(RB*3)                    END           $
2SYMX..   PH(2,2)=+PH(2,4)                                            $
2         PH(3,2)=+PH(3,4)                                            $
2         PH(4,2)=+PH(4,4)                                            $
2         PH(5,2)=+PH(5,4)                                            $
2         PH(6,2)=+PH(6,4)                                            $
2         PH(7,2)=+PH(7,4)                                            $
2         PH(8,2)=+PH(8,4)                                            $
2         PH(3,1)=+PH(3,5)                                            $
2         PH(4,1)=+PH(4,5)                                            $
2         PH(5,1)=+PH(5,5)                                            $
2         PH(6,1)=+PH(6,5)                                            $
2         PH(7,1)=+PH(7,5)                                            $
2SYMY..   PH(2,12)=+PH(2,10)                                          $
2         PH(3,12)=+PH(3,10)                                          $
2         PH(4,12)=+PH(4,10)                                          $
2         PH(5,12)=+PH(5,10)                                          $
2         PH(6,12)=+PH(6,10)                                          $
2         PH(7,12)=+PH(7,10)                                          $
2         PH(8,12)=+PH(8,10)                                          $
2         PH(3,13)=+PH(3,9)                                           $
2         PH(4,13)=+PH(4,9)                                           $
2         PH(5,13)=+PH(5,9)                                           $
2         PH(6,13)=+PH(6,9)                                           $
2         PH(7,13)=+PH(7,9)                                           $
2         C=C+1                                                       $
2         FOR J=(3,1,7)                                               $
2         FOR K=(3,1,11)                                              $
2         PH(J,K)=-(1/A3(J))(A1(J).PH(J+2,K)+A2(J).PH(J+1,K)
2         +A4(J).PH(J-1,K)+A5(J).PH(J-2,K)+A6(J).(PH(J,K+2)+PH(J,K-2))
2         +A7(J)(PH(J+1,K+1)+PH(J+1,K-1))+A8(J)(PH(J,K+1)+PH(J,K-1))
2         +A9(J)(PH(J-1,K+1)+PH(J-1,K-1))-2.0(RHO)(OMEGA*2.0)(1.0-NU))  $
2         IF C LEQ 500                                                $
2         GO TO LOOP                                                  $
```

```
2          FOR J=(2,1,8)                                                        $
2          FOR K=(3,1,11)                                                       $
2          DSQ(J,K)=((1/(M*2))+(1/(2.J(M*2))))PH(J+1,K)-((2/(M*2))
2          +(2/((J*2)(M*2)(P*2))))PH(J,K)+((1/(M*2))-(1/(2.J(M*2))))PH(J
2          -1,K)+(1/((J*2)(M*2)(P*2)))(PH(J,K+1)+PH(J,K-1))                     $
2          WRITE($$ANS,FMT)                                                     $
2INPUT     DATA(M,P,C, FOR J=(1,1,9) $ FOR K=(2,1,12) $ PH(J,K))                $
2OUTPUT    ANS(FOR J=(1,1,9) $ FOR K=(2,1,12) $ PH(J,K),FOR J=(2,1,8) $
2          FOR K=(3,1,11) $ DSQ(J,K))                                           $
2FORMAT    FMT(*5*,B2,1F15.8,W0)                                                $
2          FINISH                                                               $
```

APPENDIX C

The computer programs employed are listed in this section. Preceding each program is a list of the ALGOL symbols used and a brief explanation of their meaning.

Symbol List

Programs for ϕ_{c_1} and ϕ_d

Algol Symbol

J,K : subscripts j, k

PH(J,K) : $\phi_{j,k}$ (stress function concerned depends on program in question)

PHC1 : ϕ_{c_1}

PHD : ϕ_d

B1 through B8 : programming aids

A1(j) through A9(J) : $a_1(r)$ through $a_9(r)$

BOUND 1.. : values of stress function on boundary 1

BOUND 2.. : values of stress function on boundary 2

JPOT.. : position label of READ statement

LOOP.. : beginning point of iteration

SYMX.. : stress function values at points beyond main grid obtained from x-symmetry

SYMY.. : stress function values at points beyond main grid obtained from y-symmetry

OUT1.. : values of points outside boundary 1 obtained from $\left.\frac{\partial\phi}{\partial r}\right)_1$

OUT2.. : values of points outside boundary 2 obtained from $\left.\frac{\partial\phi}{\partial r}\right)_2$

C : number of iteration

DSQ(J,K) : $\nabla^2\phi$ (stress function concerned depends on program in question)

M : m

P : p

RHO : ρ

OMEGA : ω

NU : ν

RO : $\rho\omega^2$

RA : inner radius of disk

RB : outer radius of disk

VALUES OF STRESS FUNCTIONS

J	K	PHC1	PHD	PH
2	2	10000.000	8.0930715	511.03686
2	3	10000.000	8.0930715	511.03686
2	4	10000.000	8.0930715	511.03686
2	5	10000.000	8.0930715	511.03686
2	6	10000.000	8.0930715	511.03686
2	7	10000.000	8.0930715	511.03686
2	8	10000.000	8.0930715	511.03686
2	9	10000.000	8.0930715	511.03686
2	10	10000.000	8.0930715	511.03686
2	11	10000.000	8.0930715	511.03686
2	12	10000.000	8.0930715	511.03686
3	2	8734.1518	130.34737	569.62611
3	3	8734.1518	130.34737	569.62611
3	4	8734.1518	130.34737	569.62611
3	5	8734.1518	130.34737	569.62611
3	6	8734.1518	130.34737	569.62611
3	7	8734.1518	130.34737	569.62611
3	8	8734.1518	130.34737	569.62611
3	9	8734.1518	130.34737	569.62611
3	10	8734.1518	130.34737	569.62611
3	11	8734.1518	130.34737	569.62611
3	12	8734.1518	130.34737	569.62611
4	2	6522.5227	381.35396	709.40018
4	3	6522.5227	381.35396	709.40018
4	4	6522.5227	381.35396	709.40018
4	5	6522.5227	381.35396	709.40018
4	6	6522.5229	381.35396	709.40019
4	7	6522.5227	381.35396	709.40018
4	8	6522.5227	381.35396	709.40018
4	9	6522.5227	381.35396	709.40018
4	10	6522.5227	381.35396	709.40018
4	11	6522.5227	381.35396	709.40018
4	12	6522.5227	381.35396	709.40018
5	2	4127.5120	718.99749	926.58814

5	3	4127.5120	718.99749	926.58814
5	4	4127.5120	718.99749	926.58814
5	5	4127.5123	718.99749	926.58815
5	6	4127.5126	718.99749	926.58817
5	7	4127.5120	718.99749	926.58814
5	8	4127.5120	718.99749	926.58814
5	9	4127.5120	718.99749	926.58814
5	10	4127.5120	718.99749	926.58814
5	11	4127.5123	718.99749	926.58815
5	12	4127.5120	718.99749	926.58814
6	2	2026.1692	1120.9273	1222.8322
6	3	2026.1692	1120.9273	1222.8322
6	4	2026.1692	1120.9273	1222.8322
6	5	2026.1693	1120.9273	1222.8322
6	6	2026.1695	1120.9273	1222.8322
6	7	2026.1692	1120.9273	1222.8322
6	8	2026.1692	1120.9273	1222.8322
6	9	2026.1692	1120.9273	1222.8322
6	10	2026.1692	1120.9273	1222.8322
6	11	2026.1693	1120.9273	1222.8322
6	12	2026.1692	1120.9273	1222.8322
7	2	561.30161	1574.2809	1602.5112
7	3	561.30161	1574.2809	1602.5112
7	4	561.30161	1574.2809	1602.5112
7	5	561.30166	1574.2809	1602.5112
7	6	561.30172	1574.2809	1602.5112
7	7	561.30161	1574.2809	1602.5112
7	8	561.30161	1574.2809	1602.5112
7	9	561.30166	1574.2809	1602.5112
7	10	561.30156	1574.2809	1602.5112
7	11	561.30161	1574.2809	1602.5112
7	12	561.30156	1574.2809	1602.5112
8	2	.00000000	2071.8263	2071.8263
8	3	.00000000	2071.8263	2071.8263
8	4	.00000000	2071.8263	2071.8263
8	5	.00000000	2071.8263	2071.8263

8	6	.00000000	2071.8263	2071.8263
8	7	.00000000	2071.8263	2071.8263
8	8	.00000000	2071.8263	2071.8263
8	9	.00000000	2071.8263	2071.8263
8	10	.00000000	2071.8263	2071.8263
8	11	.00000000	2071.8263	2071.8263
8	12	.00000000	2071.8263	2071.8263

STRESSES (NUMERICAL)

J	K	RAD. STRESS	TAN. STRESS	SHEAR STRESS
2	3	-.00104300	387.78334	-.00000000
2	4	-.00104300	387.78334	-.00000000
2	5	-.00104300	387.78334	-.00000000
2	6	-.00104300	387.78334	-.00000000
2	7	-.00104300	387.78334	-.00000000
2	8	-.00104300	387.78334	-.00000000
2	9	-.00104300	387.78334	-.00000000
2	10	-.00104300	387.78334	-.00000000
2	11	-.00104300	387.78334	-.00000000
3	3	95.823500	288.32022	-.00000000
3	4	95.823500	288.32022	-.00000000
3	5	95.823500	288.32022	-.00001697
3	6	95.823500	288.32022	-.00000000
3	7	95.823500	288.32022	.00001697
3	8	95.823500	288.32022	-.00000000
3	9	95.823500	288.32022	-.00000000
3	10	95.823500	288.32022	-.00000000
3	11	95.823500	288.32022	-.00000000
4	3	113.73631	244.91106	-.00000000
4	4	113.73631	244.91106	-.00001273
4	5	113.73632	244.91106	-.00003183
4	6	113.73633	244.91106	.00001273
4	7	113.73637	244.91106	.00003183
4	8	113.73631	244.91106	-.00000000
4	9	113.73631	244.91106	-.00000000
4	10	113.73631	244.91106	-.00001273
4	11	113.73632	244.91106	-.00000000
5	3	104.20959	215.06131	-.00000000
5	4	104.20959	215.06131	.00000407
5	5	104.20919	215.06091	.00001222
5	6	104.20923	215.06095	-.00000407
5	7	104.20972	215.06131	-.00001222
5	8	104.20959	215.06131	-.00000000
5	9	104.20959	215.06131	-.00000000

5	10	104.20959	215.06131	.00000407
5	11	104.20919	215.06091	-.00000000
6	3	79.632380	188.06445	-.00000000
6	4	79.632380	188.06445	-.00000000
6	5	79.632380	188.06449	-.00000000
6	6	79.632380	188.06457	-.00000000
6	7	79.632380	188.06445	-.00000000
6	8	79.632380	188.06445	-.00000000
6	9	79.632380	188.06445	-.00000000
6	10	79.632380	188.06445	-.00000000
6	11	79.632380	188.06449	-.00000000
7	3	44.289450	160.26412	-.00000000
7	4	44.289450	160.26412	-.00000000
7	5	44.289450	160.26412	-.00000000
7	6	44.289450	160.26412	-.00000000
7	7	44.289450	160.26412	-.00000000
7	8	44.289450	160.26412	-.00000000
7	9	44.289450	160.26412	-.00000000
7	10	44.289450	160.26412	-.00000000
7	11	44.289450	160.26412	-.00000000
8	3	-.00006000	130.15327	-.00000000
8	4	-.00006000	130.15327	-.00000000
8	5	-.00006000	130.15327	-.00000000
8	6	-.00006000	130.15327	-.00000000
8	7	-.00006000	130.15327	-.00000000
8	8	-.00006000	130.15327	-.00000000
8	9	-.00006000	130.15327	-.00000000
8	10	-.00006000	130.15327	-.00000000
8	11	-.00006000	130.15327	-.00000000

STRESSES (ACTUAL)

J	RAD. STRESS	TAN. STRESS
2	-.00000000	432.97939
3	102.00643	304.67047
4	120.18213	249.67129
5	109.36573	213.14322
6	83.088890	181.55459
7	45.988065	150.26895
8	-.00000000	117.34955

Symbol List

Program for Computing Stress Distribution

Algol Symbol

DNC11 : $\dfrac{\partial\nabla^2\phi_{c_1}}{\partial n}$

DND1 : $\dfrac{\partial\nabla^2\phi_d}{\partial n}$

DSQC1 : $\nabla^2\phi_{c_1}$

DSQD : $\nabla^2\phi_{c_1}$

DSQD : $\nabla^2\phi_d$

PHC1(J,K) : $(\phi_{c_1})_{j,k}$

PHD(J,K) : $(\phi_d)_{j,k}$

PH(J,K) : $\phi_{j,k}$

SIGMAR(J,K) : $(\sigma_r)_{j,k}$ (numerical)

SIGMATH(J,K) : $(\sigma_\theta)_{j,k}$ (numerical)

TAU(J,K) : $(\zeta_{r\theta})_{j,k}$ (numerical

SIGMAR2(J) : σ_r (exact)

SIGMATH2(J) : σ_θ (exact

NU : ν

RA : Radius of path of integration

RHO : ρ

PI : π

OMEGA : ω

M : m

P : p

IV11 : $I_1 \{V\}_1$

N : Number of increments in integration path

DELX : Size of increments (radians)

F( ) : Function to be integrated

X1 : $I_1 \left\{\nabla^2 \phi_{c_1}\right\}_1$

K1 : $-I_1 \left\{\nabla^2 \phi_d\right\}_1 - (1 - \nu)\, I_1 \{V\}_1$

C1 : c_1

Q : Dummy label

V : Potential Function, V.

```
2COMMENT D.E. TATE,M.E. DEPT, NO. 1085, MASTERS THESIS,                  $
2COMMENT EVALUATION OF AIRY STRESS FUNCTION AND STRESSES IN
2        ROTATING DISK                                                   $   2
2ARRAY   DNC11(9),DND1(9),DSQC1(8,9),DSQD(8,9),PHC1(9,12),PHD(9,12),
2        PH(9,12),SIGMAR(8,11),SIGMATH(8,11),TAU(8,11),SIGMAR2(8),
2        SIGMATH2(8)                                                     $
2INTEGER I,N,J,K                                                         $
2        NU=0.30                                                         $
2        RA=1.50                                                         $
2        RHO=0.000738                                                    $
2        PI=3.14159265                                                   $
2        OMEGA=(2000.0)(8/15.0)((PI/16.0))                               $
2        M=0.50                                                          $
2        P=PI/16.0                                                       $
2        IV11=-(4.0)(PI)(RHO)(OMEGA*2)(RA*2)                             $
2PROCEDURE SIMPS(N,DELX,F())                                             $
2BEGIN   INTEGER I,N                                                     $
2        S=0.0                                                           $
2        S=F(1)+F(N+1)                                                   $
2        FOR I=(2,2,N)                                                   $
2        S=S+(4.0)(F(I))                                                 $
2        FOR I=(3,2,N-1)                                                 $
2        S=S+(2.0)(F(I))                                                 $
2        SIMPS( )=(DELX/3.0)(S)                                          $
2        RETURN                                    END SIMPS()           $
2        READ($$DATA1)                                                   $
2        READ($$DATA2)                                                   $
2        READ($$DATA3)                                                   $
2        READ($$DATA4)                                                   $
2        FOR J=(2,1,8)                                                   $
2        FOR K=(1,1,9)                                                   $
2        DSQC1(J,K)=(DSQC1(J,K))(1.0**4)                                 $
2        FOR K=(1,1,9)                                                   $
2BEGIN   DNC11(K)=(1/(2.M))(DSQC1(4,K)-DSQC1(2,K))                       $
2        DND1(K)=(1/(2.M))(DSQD(4,K)-DSQD(2,K))    END                   $
2        X1=(4.0)(RA)(SIMPS(8,P,DNC11()))                                $
```

```
2            K1=-(4.0)(RA)(SIMPS(8,P,DND1()))-(1.0-NU)(IV11)               $
2            C1=K1/X1                                                      $
2            FOR J=1                                                       $
2            FOR K=(3,1,11)                                                $
2            PHC1(J,K)=PHC1(J+2,K)                                         $
2            FOR J=9                                                       $
2            FOR K=(3,1,11)                                                $
2            PHC1(J,K)=PHC1(J-2,K)                                         $
2            FOR K=2                                                       $
2            FOR J=(1,1,9)                                                 $
2            PHC1(J,K)=PHC1(J,K+2)                                         $
2            FOR K=12                                                      $
2            FOR J=(1,1,9)                                                 $
2            PHC1(J,K)=PHC1(J,K-2)                                         $
2            FOR J=(1,1,9)                                                 $
2            FOR K=(2,1,12)                                                $
2BEGIN       PHC1(J,K)=(PHC1(J,K))(1.0**4)                                 $
2            PH(J,K)=C1.PHC1(J,K)+PHD(J,K)                  END            $
2SUBROUTINE    STRESS                                                      $
2BEGIN       Q=J                                                           $
2            V=-(0.5)(RHO)(OMEGA*2)((Q.M)*2)                               $
2            SIGMAR(J,K)=(1/((2.Q)(M*2)))(PH(J+1,K)-PH(J-1,K))
2            +(1/(((M.Q)*2)(P*2)))(PH(J,K+1)-2.PH(J,K)+PH(J,K-1))+V        $
2            SIGMATH(J,K)=(1/(M*2))(PH(J+1,K)-2.PH(J,K)+PH(J-1,K))+V       $
2            TAU(J,K)=(1/(((M.Q)*2)2.P))(PH(J,K+1)-PH(J,K-1))
2            -(1/(4.Q(M*2)P))(PH(J+1,K+1)-PH(J+1,K-1)-PH(J-1,K+1)
2            +PH(J-1,K-1))                                                 $
2            RETURN                                                        $
2            END STRESS                                                    $
2            FOR J=(2,1,8)                                                 $
2            FOR K=(3,1,11)                                                $
2            ENTER STRESS                                                  $
2            FOR J=(2,1,8)                                                 $
2BEGIN       SIGMAR2(J)=((3.0+NU)/8.0)(RHO)(OMEGA*2)(16.0+1.0
2            -(16.0/((J.M)*2))-((J.M)*2))                                  $
2            SIGMATH2(J)=((3.0+NU)/8.0)(RHO)(OMEGA*2)(16.0+1.0
```

```
2          +(16.0/((J.M)*2))-((1.0+(3.0)NU)/(3.0+NU))((J.M)*2))    END     $
2          WRITE($$ANS1,FMT1)                                                $
2          WRITE($$ANS2,FMT2)                                                $
2          WRITE($$ANS3,FMT3)                                                $
2          WRITE($$ANS4,FMT4)                                                $
2          WRITE($$ANS5,FMT5)                                                $
2INPUT     DATA1(FOR J=(2,1,8) $ FOR K=(3,1,11) $ PHC1(J,K))                 $
2INPUT     DATA2(FOR J=(2,1,8) $ FOR K=(1,1,9) $ DSQC1(J,K))                 $
2INPUT     DATA3(FOR J=(1,1,9) $ FOR K=(2,1,12) $ PHD(J,K))                  $
2INPUT     DATA4(FOR J=(2,1,8) $ FOR K=(1,1,9) $ DSQD(J,K))                  $
2OUTPUT    ANS1(FOR J=(1,1,9) $ FOR K=(2,1,12) $ BEGIN J,K, PHC1(J,K),
2          PHD(J,K)  END )                                                   $
2OUTPUT    ANS2(C1)                                                          $
2OUTPUT    ANS3(FOR J=(1,1,9) $ FOR K=(2,1,12) $ BEGIN J, K, PH(J,K)
2          END)                                                              $
2OUTPUT    ANS4(FOR J=(2,1,8) $ FOR K=(3,1,11) $ BEGIN J, K, SIGMAR(J,K),
2          SIGMATH(J,K) , TAU(J,K)  END  )                                   $
2OUTPUT    ANS5( FOR J=(2,1,8) $ BEGIN J, SIGMAR2(J), SIGMATH2(J)  END )     $
2FORMAT    FMT1(B41,2I3,2S16.8,W6)                                           $
2FORMAT    FMT2(B52,1S16.8,W6)                                               $
2FORMAT    FMT3(B49,2I3,1S16.8,W6)                                           $
2FORMAT    FMT4(B30,2I3,3S16.8,W6)                                           $
2FORMAT    FMT5(B33,1I3,2S16.8,W6)                                           $
2          FINISH                                                            $
```

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